MULTIMEDIA UNIVERSITY®

STUDENT ID NO

MULTIMEDIA UNIVERSITY

FINAL EXAMINATION

TRIMESTER 3, 2022/2023 (TERM ID 2230)

EMT2036 ENGINEERING MATHEMATICS 3

(All Sections / Groups)

2 OCTOBER 2023
09:00 A.M – 11:00 A.M.
(2 Hours)

INSTRUCTIONS TO STUDENTS

1. This question paper consists of 15 pages including this cover page with 5 questions only.

2. Attempt all 5 questions. The questions do not have equal mark distribution. The distribution of marks for each question is given.

3. Please print all your answers in the answer booklet provided.

Question 1 (Statistics)

NOTE: You may refer to **Appendix B** to solve this question.

A machine produces rubber balls. The distribution of the balls' diameter is known to be approximately normal. In a recent quality control inspection, the diameters of 10 randomly selected balls are measured. The results in millimetres (mm) are given in Table Q1.

(a) Estimate the 95% confidence interval for the diameters' mean μ.

[6 marks]

(b) Perform a hypothesis test at the 0.05 level of significance to check if the diameters' mean is greater than the design value, i.e. 200 mm.

[6 marks]

Table Q1

205	211	202	196	196
201	208	193	203	205

Continued ...

Question 2 (Multiple Integrals)

Evaluate the double integrals

$$\iint_S \frac{x}{\sqrt{x^2+y^2}}\, dS$$

over the region S defined by $x^2 + y^2 \leq 2y$ and $x \geq 0$, as shown in Fig. Q2 with S being shaded.

[13 marks]

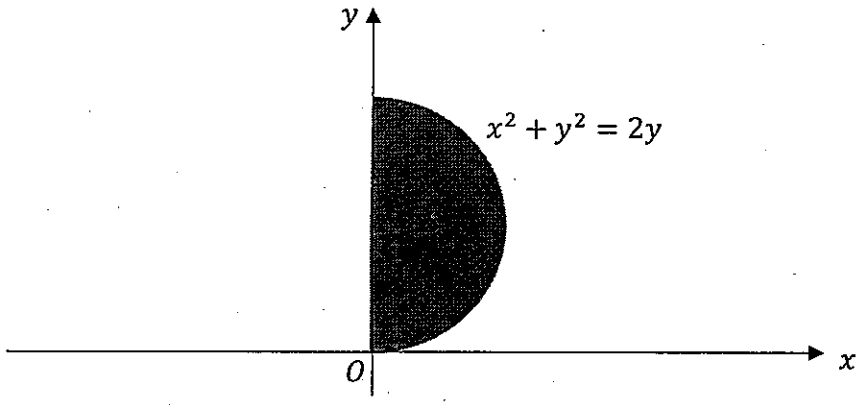


Fig. Q2

Continued ...

Question 3 (Vector Calculus)

(a) Evaluate the surface integral

$$\iint_W (x\,\hat{\imath} + \ln(1+z)\,\hat{k}) \cdot \hat{n}\, dW$$

where W is one-half of the curved surface of a circular cylinder of radius 5 centred at the z-axis, and $\hat{n}$ is the unit normal of W, as shown in Fig. Q3-1.

[12 marks]

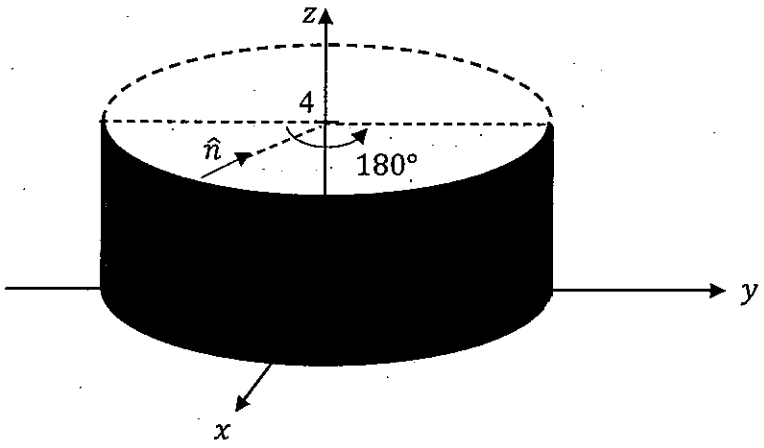


Fig. Q3-1

(b) Evaluate the closed surface integral

$$\oiint_A (2x\,\hat{\imath} + y\hat{\jmath} + 2\hat{k}) \cdot \hat{n}\, dA$$

where A is the closed surface bounding a spherical cone defined by $\sqrt{x^2+y^2} \leq z \leq \sqrt{4-x^2-y^2}$, and $\hat{n}$ is the outward unit normal of A, as shown in Fig. Q3-2.

[13 marks]

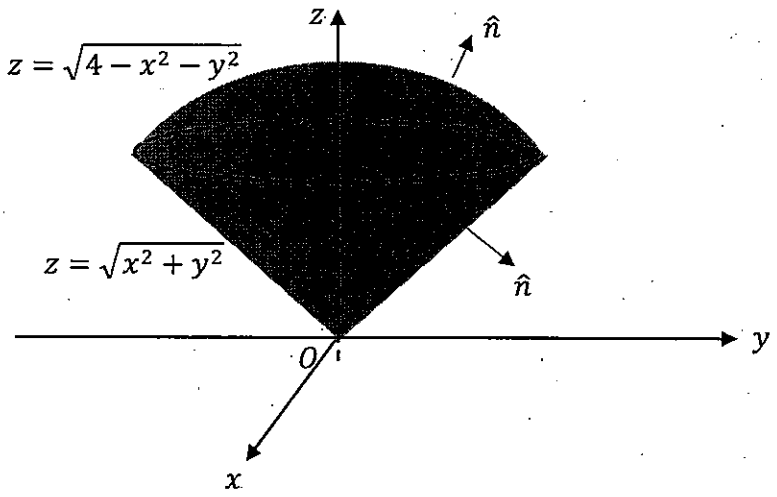


Fig. Q3-2

Continued ...

Question 4 (Linear Algebra)

(a) Given

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = [2 \quad 1 \quad 3],$$

compute AB or BA, whichever is applicable.

[2 marks]

(b) Given

$$C = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix},$$

answer the following:

(i) Compute the inverse of C.

[4 marks]

(ii) Write C as a sum of a symmetric matrix and a skew symmetric matrix.

[3 marks]

(iii) Write C as a sum of two strictly triangular matrices and one diagonal matrix.

[3 marks]

(c) Write down the 3×3 elementary matrices for the following elementary row operations:

(i) Swap Row #1 and Row #3.
(ii) Multiply Row #2 by scalar 2.
(iii) Add 3 times Row #1 to Row #3.

[3 marks]

(d) Solve the following system of linear equations using Cramer's rule.

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ x_2 + 2x_3 &= 3 \\ 3x_1 - x_2 - x_3 &= 2 \end{aligned}$$

[10 marks]

Continued ...

Question 5 (Linear Algebra)

Consider the following system of linear differential equations:

$$\begin{gathered} x_1' - 3x_1 - 2x_2 + 2x_3 = 4 + 3t \\ x_2' + 2x_1 + 5x_2 - 4x_3 = 2 \\ x_3' + x_1 + 5x_2 - 4x_3 = 4 + 2t \\ x_1(0) = 1, \qquad x_2(0) = 0, \qquad x_3(0) = 2 \end{gathered}$$

where x_1, x_2, and x_3 are functions of t and $x' = \frac{dx}{dt}$.

(a) The above system may be decoupled with a suitable substitution of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{P} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Find the matrix **P**.

[17 marks]

(b) With the substitution in (a), determine the resultant system of decoupled linear differential equations including the initial conditions. You are **not** required to solve the differential equations.

[8 marks]

End of Paper

Appendix A: Coordinate Systems' Vector Basis

Transformation	Vector basis
Cartesian & Cylindrical	$\hat{r} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta$ $\hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$ $\hat{k} = \hat{k}$
	$\hat{\imath} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$ $\hat{\jmath} = \hat{r}\sin\theta + \hat{\theta}\cos\theta$ $\hat{k} = \hat{k}$
Spherical & Cartesian	$\hat{\rho} = \hat{\imath}\sin\phi\cos\theta + \hat{\jmath}\sin\phi\sin\theta + \hat{k}\cos\phi$ $\hat{\phi} = \hat{\imath}\cos\phi\cos\theta + \hat{\jmath}\cos\phi\sin\theta - \hat{k}\sin\phi$ $\hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$
	$\hat{\imath} = \hat{\rho}\sin\phi\cos\theta + \hat{\phi}\cos\phi\cos\theta - \hat{\theta}\sin\theta$ $\hat{\jmath} = \hat{\rho}\sin\phi\sin\theta + \hat{\phi}\cos\phi\sin\theta + \hat{\theta}\cos\theta$ $\hat{k} = \hat{\rho}\cos\phi - \hat{\phi}\sin\phi$
Cylindrical & Spherical	$\hat{\rho} = \hat{r}\sin\phi + \hat{k}\cos\phi$ $\hat{\phi} = \hat{r}\cos\phi - \hat{k}\sin\phi$ $\hat{\theta} = \hat{\theta}$
	$\hat{r} = \hat{\rho}\sin\phi + \hat{\phi}\cos\phi$ $\hat{\theta} = \hat{\theta}$ $\hat{k} = \hat{\rho}\cos\phi - \hat{\phi}\sin\phi$

APPENDIX B: STATISTICAL FORMULAE & TABLES

1. Important Statistics

a) Sample mean: $\overline{X} = \sum_{i=1}^{n} \frac{X_i}{n}$

b) Sample variance: $S^2 = \sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{n-1}$ or $\frac{n\sum_{i=1}^{n} X_i^2 - \left(\sum_{i=1}^{n} X_i\right)^2}{n(n-1)}$

c) Sample proportion: $\hat{p} = \frac{x}{n}$

2. Sampling Distributions

a) Sampling distributions of $\overline{X}$

Sample size, n	Large ($n \geq 30$)		Small ($n < 30$)	
Population distribution	Any		Normal	
σ^2 known?	Y	N	Y	N
Sample mean distribution	$\overline{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$	$\overline{X} \sim N\left(\mu, \frac{S}{\sqrt{n}}\right)$	$\overline{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$	$T \sim T(\upsilon = n-1)$ where $T = \frac{\overline{X} - \mu}{S/\sqrt{n}}$

b) Sampling distribution of S^2: $\chi^2 \sim \chi^2(\upsilon = n-1)$, where $\chi^2 = \frac{(n-1)S^2}{\sigma^2}$.

Remark: Only valid for normal population.

c) Sampling distribution of $\hat{p}$: $\hat{p} \sim N\left(p, \sqrt{\frac{pq}{n}}\right)$.

Remark: Only valid for large samples.

3. Confidence Interval for Population Mean μ

Scenario	Two-sided C.I.	One-sided C.I. bounded from above	One-sided C.I. bounded from below
Large sample from any population	$\overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$	$\mu < \overline{x} + z_{\alpha}\frac{\sigma}{\sqrt{n}}$	$\mu > \overline{x} - z_{\alpha}\frac{\sigma}{\sqrt{n}}$
	Remark: Replace σ with s if σ^2 is unknown.		
Small sample from normal population, σ^2 known	$\overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$	$\mu < \overline{x} + z_{\alpha}\frac{\sigma}{\sqrt{n}}$	$\mu > \overline{x} - z_{\alpha}\frac{\sigma}{\sqrt{n}}$
Small sample from normal population, σ^2 unknown	$\overline{x} - t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \overline{x} + t_{\alpha/2}\frac{s}{\sqrt{n}}$	$\mu < \overline{x} + t_{\alpha}\frac{s}{\sqrt{n}}$	$\mu > \overline{x} - t_{\alpha}\frac{s}{\sqrt{n}}$

4. **Confidence Interval for Population Proportion p:** $\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}} < p < \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}}$

5. **Confidence Interval for PopulationVariance σ^2:** $\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}$

6. **Hypothesis Testing of a Population Mean μ**

a) Large sample from any population, σ^2 known or unknown

H_0:	$\mu = \mu_0$	$\mu = \mu_0$	$\mu = \mu_0$
H_1:	$\mu > \mu_0$	$\mu < \mu_0$	$\mu \neq \mu_0$
Test distribution	Standard Normal		
Test statistic	$z = \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}}$		
Critical region	$z > z_\alpha$	$z < -z_\alpha$	$z > z_{\alpha/2}$ or $z < -z_{\alpha/2}$
***p*-value**	$P(Z > z)$	$P(Z < z)$	$P(Z > \lvert z\rvert) + P(Z < -\lvert z\rvert)$

Remark: Replace σ with s if σ^2 is unknown.

b) Small sample from normal population, σ^2 known
Refer to the table in a).

c) Small sample from normal population, σ^2 unknown

H_0:	$\mu = \mu_0$	$\mu = \mu_0$	$\mu = \mu_0$
H_1:	$\mu > \mu_0$	$\mu < \mu_0$	$\mu \neq \mu_0$
Test distribution	t-distribution with $\upsilon = n-1$ degrees of freedom		
Test statistic	$t = \frac{\overline{x} - \mu_0}{s/\sqrt{n}}$		
Critical region	$t > t_\alpha$	$t < -t_\alpha$	$t > t_{\alpha/2}$ or $t < -t_{\alpha/2}$

7. **Hypothesis Testing of Population Proportion p**

H_0:	$p = p_0$	$p = p_0$	$p = p_0$
H_1:	$p > p_0$	$p < p_0$	$p \neq p_0$
Test distribution	Standard Normal		
Test statistic	$z = \frac{\hat{p} - p_0}{\sqrt{p_0 q_0 / n}}$ where $\hat{p} = x/n$ and $q_0 = 1 - p_0$.		
Critical region	$z > z_\alpha$	$z < -z_\alpha$	$z > z_{\alpha/2}$ or $z < -z_{\alpha/2}$
***p*-value**	$P(Z > z)$	$P(Z < z)$	$P(Z > \lvert z\rvert) + P(Z < -\lvert z\rvert)$

8. Hypothesis Testing of Population Variance σ^2

H_0 H_1	$\sigma^2 = \sigma_0^2$ $\sigma^2 > \sigma_0^2$	$\sigma^2 = \sigma_0^2$ $\sigma^2 < \sigma_0^2$	$\sigma^2 = \sigma_0^2$ $\sigma^2 \neq \sigma_0^2$
Test distribution	χ^2-distribution with $\upsilon = n-1$ degrees of freedom		
Test statistic	$\chi^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$		
Critical region	$\chi^2 > \chi_\alpha^2$	$\chi^2 < \chi_{1-\alpha}^2$	$\chi^2 > \chi_{\alpha/2}^2$ or $\chi^2 < \chi_{1-\alpha/2}^2$

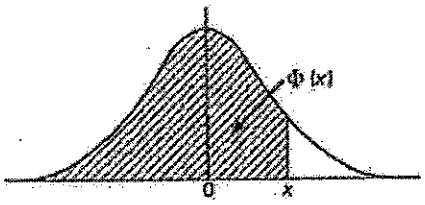
9. Chi-squared Test of Goodness of Fit

H_0 H_1	the population follows probability distribution $f(x)$ the population does not follow probability distribution $f(x)$
Test distribution	χ^2-distribution with $\upsilon = k-1$ degrees of freedom
Test statistic	$\chi^2 = \sum_{i=1}^{k} \dfrac{(o_i - e_i)^2}{e_i}$ where o_i = observed frequency, e_i = expected frequency and k = number of classes.
Critical region	$\chi^2 > \chi_\alpha^2$

Remark: e_i must be at least 5 units.

TABLE 4. THE NORMAL DISTRIBUTION FUNCTION

The function tabulated is $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt$. $\Phi(x)$ is the probability that a random variable, normally distributed with zero mean and unit variance, will be less than or equal to x. When $x < 0$ use $\Phi(x) = 1 - \Phi(-x)$, as the normal distribution with zero mean and unit variance is symmetric about zero.



x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0·00	0·5000	0·40	0·6554	0·80	0·7881	1·20	0·8849	1·60	0·9452	2·00	0·97725
·01	·5040	·41	·6591	·81	·7910	·21	·8869	·61	·9463	·01	·97778
·02	·5080	·42	·6628	·82	·7939	·22	·8888	·62	·9474	·02	·97831
·03	·5120	·43	·6664	·83	·7967	·23	·8907	·63	·9484	·03	·97882
·04	·5160	·44	·6700	·84	·7995	·24	·8925	·64	·9495	·04	·97932
0·05	0·5199	0·45	0·6736	0·85	0·8023	1·25	0·8944	1·65	0·9505	2·05	0·97982
·06	·5239	·46	·6772	·86	·8051	·26	·8962	·66	·9515	·06	·98030
·07	·5279	·47	·6808	·87	·8078	·27	·8980	·67	·9525	·07	·98077
·08	·5319	·48	·6844	·88	·8106	·28	·8997	·68	·9535	·08	·98124
·09	·5359	·49	·6879	·89	·8133	·29	·9015	·69	·9545	·09	·98169
0·10	0·5398	0·50	0·6915	0·90	0·8159	1·30	0·9032	1·70	0·9554	2·10	0·98214
·11	·5438	·51	·6950	·91	·8186	·31	·9049	·71	·9564	·11	·98257
·12	·5478	·52	·6985	·92	·8212	·32	·9066	·72	·9573	·12	·98300
·13	·5517	·53	·7019	·93	·8238	·33	·9082	·73	·9582	·13	·98341
·14	·5557	·54	·7054	·94	·8264	·34	·9099	·74	·9591	·14	·98382
0·15	0·5596	0·55	0·7088	0·95	0·8289	1·35	0·9115	1·75	0·9599	2·15	0·98422
·16	·5636	·56	·7123	·96	·8315	·36	·9131	·76	·9608	·16	·98461
·17	·5675	·57	·7157	·97	·8340	·37	·9147	·77	·9616	·17	·98500
·18	·5714	·58	·7190	·98	·8365	·38	·9162	·78	·9625	·18	·98537
·19	·5753	·59	·7224	·99	·8389	·39	·9177	·79	·9633	·19	·98574
0·20	0·5793	0·60	0·7257	1·00	0·8413	1·40	0·9192	1·80	0·9641	2·20	0·98610
·21	·5832	·61	·7291	·01	·8438	·41	·9207	·81	·9649	·21	·98645
·22	·5871	·62	·7324	·02	·8461	·42	·9222	·82	·9656	·22	·98679
·23	·5910	·63	·7357	·03	·8485	·43	·9236	·83	·9664	·23	·98713
·24	·5948	·64	·7389	·04	·8508	·44	·9251	·84	·9671	·24	·98745
0·25	0·5987	0·65	0·7422	1·05	0·8531	1·45	0·9265	1·85	0·9678	2·25	0·98778
·26	·6026	·66	·7454	·06	·8554	·46	·9279	·86	·9686	·26	·98809
·27	·6064	·67	·7486	·07	·8577	·47	·9292	·87	·9693	·27	·98840
·28	·6103	·68	·7517	·08	·8599	·48	·9306	·88	·9699	·28	·98870
·29	·6141	·69	·7549	·09	·8621	·49	·9319	·89	·9706	·29	·98899
0·30	0·6179	0·70	0·7580	1·10	0·8643	1·50	0·9332	1·90	0·9713	2·30	0·98928
·31	·6217	·71	·7611	·11	·8665	·51	·9345	·91	·9719	·31	·98956
·32	·6255	·72	·7642	·12	·8686	·52	·9357	·92	·9726	·32	·98983
·33	·6293	·73	·7673	·13	·8708	·53	·9370	·93	·9732	·33	·99010
·34	·6331	·74	·7704	·14	·8729	·54	·9382	·94	·9738	·34	·99036
0·35	0·6368	0·75	0·7734	1·15	0·8749	1·55	0·9394	1·95	0·9744	2·35	0·99061
·36	·6406	·76	·7764	·16	·8770	·56	·9406	·96	·9750	·36	·99086
·37	·6443	·77	·7794	·17	·8790	·57	·9418	·97	·9756	·37	·99111
·38	·6480	·78	·7823	·18	·8810	·58	·9429	·98	·9761	·38	·99134
·39	·6517	·79	·7852	·19	·8830	·59	·9441	·99	·9767	·39	·99158
0·40	0·6554	0·80	0·7881	1·20	0·8849	1·60	0·9452	2·00	0·9772	2·40	0·99180

TABLE 4. THE NORMAL DISTRIBUTION FUNCTION

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
2·40	0·99180	2·55	0·99461	2·70	0·99653	2·85	0·99781	3·00	0·99865	3·15	0·99918
·41	·99202	·56	·99477	·71	·99664	·86	·99788	·01	·99869	·16	·99921
·42	·99224	·57	·99492	·72	·99674	·87	·99795	·02	·99874	·17	·99924
·43	·99245	·58	·99506	·73	·99683	·88	·99801	·03	·99878	·18	·99926
·44	·99266	·59	·99520	·74	·99693	·89	·99807	·04	·99882	·19	·99929
2·45	0·99286	2·60	0·99534	2·75	0·99702	2·90	0·99813	3·05	0·99886	3·20	0·99931
·46	·99305	·61	·99547	·76	·99711	·91	·99819	·06	·99889	·21	·99934
·47	·99324	·62	·99560	·77	·99720	·92	·99825	·07	·99893	·22	·99936
·48	·99343	·63	·99573	·78	·99728	·93	·99831	·08	·99896	·23	·99938
·49	·99361	·64	·99585	·79	·99736	·94	·99836	·09	·99900	·24	·99940
2·50	0·99379	2·65	0·99598	2·80	0·99744	2·95	0·99841	3·10	0·99903	3·25	0·99942
·51	·99396	·66	·99609	·81	·99752	·96	·99846	·11	·99906	·26	·99944
·52	·99413	·67	·99621	·82	·99760	·97	·99851	·12	·99910	·27	·99946
·53	·99430	·68	·99632	·83	·99767	·98	·99856	·13	·99913	·28	·99948
·54	·99446	·69	·99643	·84	·99774	·99	·99861	·14	·99916	·29	·99950
2·55	0·99461	2·70	0·99653	2·85	0·99781	3·00	0·99865	3·15	0·99918	3·30	0·99952

The critical table below gives on the left the range of values of x for which $\Phi(x)$ takes the value on the right, correct to the last figure given; in critical cases, take the upper of the two values of $\Phi(x)$ indicated.

3·075	0·9990	3·263	0·9994	3·731	0·99990	3·916	0·99995
3·105	0·9991	3·320	0·9995	3·742	0·99991	3·976	0·99996
3·138	0·9992	3·389	0·9996	3·759	0·99992	4·055	0·99997
3·174	0·9993	3·480	0·9997	3·781	0·99993	4·173	0·99998
3·215	0·9994	3·615	0·9998	3·826	0·99994	4·417	0·99999
			0·9999	3·867	0·99995		1·00000

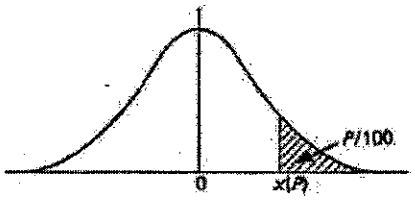
When $x > 3{\cdot}3$ the formula $1-\Phi(x) = \dfrac{e^{-\frac{1}{2}x^2}}{x\sqrt{2\pi}}\left[1-\dfrac{1}{x^2}+\dfrac{3}{x^4}-\dfrac{15}{x^6}+\dfrac{105}{x^8}\right]$ is very accurate, with relative error less than $945/x^{10}$.

TABLE 5. PERCENTAGE POINTS OF THE NORMAL DISTRIBUTION

This table gives percentage points $x(P)$ defined by the equation

$$\frac{P}{100} = \frac{1}{\sqrt{2\pi}}\int_{x(P)}^{\infty} e^{-\frac{1}{2}t^2}\,dt.$$

If X is a variable, normally distributed with zero mean and unit variance, $P/100$ is the probability that $X \geqslant x(P)$. The lower P per cent points are given by symmetry as $-x(P)$, and the probability that $|X| \geqslant x(P)$ is $2P/100$.



P	$x(P)$	P	$x(P)$	P	$x(P)$	P	$x(P)$	P	$x(P)$	P	$x(P)$
50	0·0000	5·0	1·6449	3·0	1·8808	2·0	2·0537	1·0	2·3263	0·10	3·0902
45	0·1257	4·8	1·6646	2·9	1·8957	1·9	2·0749	0·9	2·3656	0·09	3·1214
40	0·2533	4·6	1·6849	2·8	1·9110	1·8	2·0969	0·8	2·4089	0·08	3·1559
35	0·3853	4·4	1·7060	2·7	1·9268	1·7	2·1201	0·7	2·4573	0·07	3·1947
30	0·5244	4·2	1·7279	2·6	1·9431	1·6	2·1444	0·6	2·5121	0·06	3·2389
25	0·6745	4·0	1·7507	2·5	1·9600	1·5	2·1701	0·5	2·5758	0·05	3·2905
20	0·8416	3·8	1·7744	2·4	1·9774	1·4	2·1973	0·4	2·6521	0·01	3·7190
15	1·0364	3·6	1·7991	2·3	1·9954	1·3	2·2262	0·3	2·7478	0·005	3·8906
10	1·2816	3·4	1·8250	2·2	2·0141	1·2	2·2571	0·2	2·8782	0·001	4·2649
5	1·6449	3·2	1·8522	2·1	2·0335	1·1	2·2904	0·1	3·0902	0·0005	4·4172

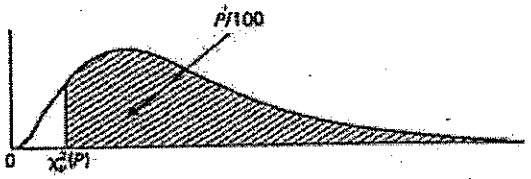
TABLE 8. PERCENTAGE POINTS OF THE χ^2-DISTRIBUTION

This table gives percentage points $\chi^2_\nu(P)$ defined by the equation

$$\frac{P}{100} = \frac{1}{2^{\nu/2}\,\Gamma(\frac{\nu}{2})} \int_{\chi^2(P)}^{\infty} x^{\frac{1}{2}\nu - 1}\, e^{-\frac{1}{2}x}\, dx.$$

If X is a variable distributed as χ^2 with ν degrees of freedom, $P/100$ is the probability that $X \geqslant \chi^2_\nu(P)$.

For $\nu > 100$, $\sqrt{2X}$ is approximately normally distributed with mean $\sqrt{2\nu - 1}$ and unit variance.



(The above shape applies for $\nu \geqslant 3$ only. When $\nu < 3$ the mode is at the origin.)

P	99·95	99·9	99·5	99	97·5	95	90	80	70	60
$\nu = 1$	$0{\cdot}0^5 3927$	$0{\cdot}0^4 1571$	$0{\cdot}0^4 3927$	$0{\cdot}0^3 1571$	$0{\cdot}0^3 9821$	0·003932	0·01579	0·06418	0·1485	0·2750
2	0·001000	0·002001	0·01003	0·02010	0·05064	0·1026	0·2107	0·4463	0·7133	1·022
3	0·01528	0·02430	0·07172	0·1148	0·2158	0·3518	0·5844	1·005	1·424	1·869
4	0·06392	0·09080	0·2070	0·2971	0·4844	0·7107	1·064	1·649	2·195	2·753
5	0·1581	0·2102	0·4117	0·5543	0·8312	1·145	1·610	2·343	3·000	3·655
6	0·2994	0·3811	0·6757	0·8721	1·237	1·635	2·204	3·070	3·828	4·570
7	0·4849	0·5985	0·9893	1·239	1·690	2·167	2·833	3·822	4·671	5·493
8	0·7104	0·8571	1·344	1·646	2·180	2·733	3·490	4·594	5·527	6·423
9	0·9717	1·152	1·735	2·088	2·700	3·325	4·168	5·380	6·393	7·357
10	1·265	1·479	2·156	2·558	3·247	3·940	4·865	6·179	7·267	8·295
11	1·587	1·834	2·603	3·053	3·816	4·575	5·578	6·989	8·148	9·237
12	1·934	2·214	3·074	3·571	4·404	5·226	6·304	7·807	9·034	10·18
13	2·305	2·617	3·565	4·107	5·009	5·892	7·042	8·634	9·926	11·13
14	2·697	3·041	4·075	4·660	5·629	6·571	7·790	9·467	10·82	12·08
15	3·108	3·483	4·601	5·229	6·262	7·261	8·547	10·31	11·72	13·03
16	3·536	3·942	5·142	5·812	6·908	7·962	9·312	11·15	12·62	13·98
17	3·980	4·416	5·697	6·408	7·564	8·672	10·09	12·00	13·53	14·94
18	4·439	4·905	6·265	7·015	8·231	9·390	10·86	12·86	14·44	15·89
19	4·912	5·407	6·844	7·633	8·907	10·12	11·65	13·72	15·35	16·85
20	5·398	5·921	7·434	8·260	9·591	10·85	12·44	14·58	16·27	17·81
21	5·896	6·447	8·034	8·897	10·28	11·59	13·24	15·44	17·18	18·77
22	6·404	6·983	8·643	9·542	10·98	12·34	14·04	16·31	18·10	19·73
23	6·924	7·529	9·260	10·20	11·69	13·09	14·85	17·19	19·02	20·69
24	7·453	8·085	9·886	10·86	12·40	13·85	15·66	18·06	19·94	21·65
25	7·991	8·649	10·52	11·52	13·12	14·61	16·47	18·94	20·87	22·62
26	8·538	9·222	11·16	12·20	13·84	15·38	17·29	19·82	21·79	23·58
27	9·093	9·803	11·81	12·88	14·57	16·15	18·11	20·70	22·72	24·54
28	9·656	10·39	12·46	13·56	15·31	16·93	18·94	21·59	23·65	25·51
29	10·23	10·99	13·12	14·26	16·05	17·71	19·77	22·48	24·58	26·48
30	10·80	11·59	13·79	14·95	16·79	18·49	20·60	23·36	25·51	27·44
32	11·98	12·81	15·13	16·36	18·29	20·07	22·27	25·15	27·37	29·38
34	13·18	14·06	16·50	17·79	19·81	21·66	23·95	26·94	29·24	31·31
36	14·40	15·32	17·89	19·23	21·34	23·27	25·64	28·73	31·12	33·25
38	15·64	16·61	19·29	20·69	22·88	24·88	27·34	30·54	32·99	35·19
40	16·91	17·92	20·71	22·16	24·43	26·51	29·05	32·34	34·87	37·13
50	23·46	24·67	27·99	29·71	32·36	34·76	37·69	41·45	44·31	46·86
60	30·34	31·74	35·53	37·48	40·48	43·19	46·46	50·64	53·81	56·62
70	37·47	39·04	43·28	45·44	48·76	51·74	55·33	59·90	63·35	66·40
80	44·79	46·52	51·17	53·54	57·15	60·39	64·28	69·21	72·92	76·19
90	52·28	54·16	59·20	61·75	65·65	69·13	73·29	78·56	82·51	85·99
100	59·90	61·92	67·33	70·06	74·22	77·93	82·36	87·95	92·13	95·81

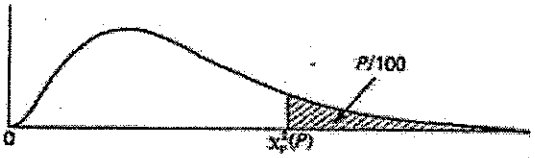
TABLE 8. PERCENTAGE POINTS OF THE χ^2-DISTRIBUTION

This table gives percentage points $\chi^2_\nu(P)$ defined by the equation

$$\frac{P}{100} = \frac{1}{2^{\nu/2}\,\Gamma(\frac{\nu}{2})} \int_{\chi^2_\nu(P)}^{\infty} x^{\frac{1}{2}\nu - 1} e^{-\frac{1}{2}x}\, dx.$$

If X is a variable distributed as χ^2 with ν degrees of freedom, $P/100$ is the probability that $X \geqslant \chi^2_\nu(P)$.

For $\nu > 100$, $\sqrt{2X}$ is approximately normally distributed with mean $\sqrt{2\nu - 1}$ and unit variance.



(The above shape applies for $\nu \geqslant 3$ only. When $\nu < 3$ the mode is at the origin.)

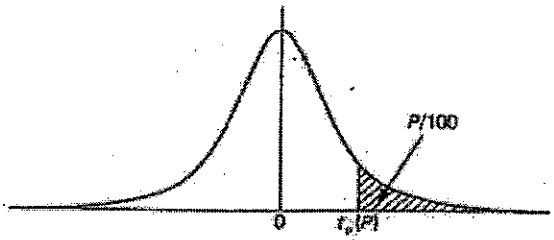
P	50	40	30	20	10	5	2·5	1	0·5	0·1	0·05
$\nu = 1$	0·4549	0·7083	1·074	1·642	2·706	3·841	5·024	6·635	7·879	10·83	12·12
2	1·386	1·833	2·408	3·219	4·605	5·991	7·378	9·210	10·60	13·82	15·20
3	2·366	2·946	3·665	4·642	6·251	7·815	9·348	11·34	12·84	16·27	17·73
4	3·357	4·045	4·878	5·989	7·779	9·488	11·14	13·28	14·86	18·47	20·00
5	4·351	5·132	6·064	7·289	9·236	11·07	12·83	15·09	16·75	20·52	22·11
6	5·348	6·211	7·231	8·558	10·64	12·59	14·45	16·81	18·55	22·46	24·10
7	6·346	7·283	8·383	9·803	12·02	14·07	16·01	18·48	20·28	24·32	26·02
8	7·344	8·351	9·524	11·03	13·36	15·51	17·53	20·09	21·95	26·12	27·87
9	8·343	9·414	10·66	12·24	14·68	16·92	19·02	21·67	23·59	27·88	29·67
10	9·342	10·47	11·78	13·44	15·99	18·31	20·48	23·21	25·19	29·59	31·42
11	10·34	11·53	12·90	14·63	17·28	19·68	21·92	24·72	26·76	31·26	33·14
12	11·34	12·58	14·01	15·81	18·55	21·03	23·34	26·22	28·30	32·91	34·82
13	12·34	13·64	15·12	16·98	19·81	22·36	24·74	27·69	29·82	34·53	36·48
14	13·34	14·69	16·22	18·15	21·06	23·68	26·12	29·14	31·32	36·12	38·11
15	14·34	15·73	17·32	19·31	22·31	25·00	27·49	30·58	32·80	37·70	39·72
16	15·34	16·78	18·42	20·47	23·54	26·30	28·85	32·00	34·27	39·25	41·31
17	16·34	17·82	19·51	21·61	24·77	27·59	30·19	33·41	35·72	40·79	42·88
18	17·34	18·87	20·60	22·76	25·99	28·87	31·53	34·81	37·16	42·31	44·43
19	18·34	19·91	21·69	23·90	27·20	30·14	32·85	36·19	38·58	43·82	45·97
20	19·34	20·95	22·77	25·04	28·41	31·41	34·17	37·57	40·00	45·31	47·50
21	20·34	21·99	23·86	26·17	29·62	32·67	35·48	38·93	41·40	46·80	49·01
22	21·34	23·03	24·94	27·30	30·81	33·92	36·78	40·29	42·80	48·27	50·51
23	22·34	24·07	26·02	28·43	32·01	35·17	38·08	41·64	44·18	49·73	52·00
24	23·34	25·11	27·10	29·55	33·20	36·42	39·36	42·98	45·56	51·18	53·48
25	24·34	26·14	28·17	30·68	34·38	37·65	40·65	44·31	46·93	52·62	54·95
26	25·34	27·18	29·25	31·79	35·56	38·89	41·92	45·64	48·29	54·05	56·41
27	26·34	28·21	30·32	32·91	36·74	40·11	43·19	46·96	49·64	55·48	57·86
28	27·34	29·25	31·39	34·03	37·92	41·34	44·46	48·28	50·99	56·89	59·30
29	28·34	30·28	32·46	35·14	39·09	42·56	45·72	49·59	52·34	58·30	60·73
30	29·34	31·32	33·53	36·25	40·26	43·77	46·98	50·89	53·67	59·70	62·16
32	31·34	33·38	35·66	38·47	42·58	46·19	49·48	53·49	56·33	62·49	65·00
34	33·34	35·44	37·80	40·68	44·90	48·60	51·97	56·06	58·96	65·25	67·80
36	35·34	37·50	39·92	42·88	47·21	51·00	54·44	58·62	61·58	67·99	70·59
38	37·34	39·56	42·05	45·08	49·51	53·38	56·90	61·16	64·18	70·70	73·35
40	39·34	41·62	44·16	47·27	51·81	55·76	59·34	63·69	66·77	73·40	76·09
50	49·33	51·89	54·72	58·16	63·17	67·50	71·42	76·15	79·49	86·66	89·56
60	59·33	62·13	65·23	68·97	74·40	79·08	83·30	88·38	91·95	99·61	102·7
70	69·33	72·36	75·69	79·71	85·53	90·53	95·02	100·4	104·2	112·3	115·6
80	79·33	82·57	86·12	90·41	96·58	101·9	106·6	112·3	116·3	124·8	128·3
90	89·33	92·76	96·52	101·1	107·6	113·1	118·1	124·1	128·3	137·2	140·8
100	99·33	102·9	106·9	111·7	118·5	124·3	129·6	135·8	140·2	149·4	153·2

TABLE 10. PERCENTAGE POINTS OF THE t-DISTRIBUTION

This table gives percentage points $t_\nu(P)$ defined by the equation

$$\frac{P}{100} = \frac{1}{\sqrt{\nu\pi}} \frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2})}{\Gamma(\frac{1}{2}\nu)} \int_{t_\nu(P)}^{\infty} \frac{dt}{(1+t^2/\nu)^{\frac{1}{2}(\nu+1)}}.$$

Let X_1 and X_2 be independent random variables having a normal distribution with zero mean and unit variance and a χ^2-distribution with ν degrees of freedom respectively; then $t = X_1/\sqrt{X_2/\nu}$ has Student's t-distribution with ν degrees of freedom, and the probability that $t \geqslant t_\nu(P)$ is $P/100$. The lower percentage points are given by symmetry as $-t_\nu(P)$, and the probability that $|t| \geqslant t_\nu(P)$ is $2P/100$.



The limiting distribution of t as ν tends to infinity is the normal distribution with zero mean and unit variance. When ν is large interpolation in ν should be harmonic.

P	40	30	25	20	15	10	5	2·5	1	0·5	0·1	0·05
$\nu = 1$	0·3249	0·7265	1·0000	1·3764	1·963	3·078	6·314	12·71	31·82	63·66	318·3	636·6
2	0·2887	0·6172	0·8165	1·0607	1·386	1·886	2·920	4·303	6·965	9·925	22·33	31·60
3	0·2767	0·5844	0·7649	0·9785	1·250	1·638	2·353	3·182	4·541	5·841	10·21	12·92
4	0·2707	0·5686	0·7407	0·9410	1·190	1·533	2·132	2·776	3·747	4·604	7·173	8·610
5	0·2672	0·5594	0·7267	0·9195	1·156	1·476	2·015	2·571	3·365	4·032	5·893	6·869
6	0·2648	0·5534	0·7176	0·9057	1·134	1·440	1·943	2·447	3·143	3·707	5·208	5·959
7	0·2632	0·5491	0·7111	0·8960	1·119	1·415	1·895	2·365	2·998	3·499	4·785	5·408
8	0·2619	0·5459	0·7064	0·8889	1·108	1·397	1·860	2·306	2·896	3·355	4·501	5·041
9	0·2610	0·5435	0·7027	0·8834	1·100	1·383	1·833	2·262	2·821	3·250	4·297	4·781
10	0·2602	0·5415	0·6998	0·8791	1·093	1·372	1·812	2·228	2·764	3·169	4·144	4·587
11	0·2596	0·5399	0·6974	0·8755	1·088	1·363	1·796	2·201	2·718	3·106	4·025	4·437
12	0·2590	0·5386	0·6955	0·8726	1·083	1·356	1·782	2·179	2·681	3·055	3·930	4·318
13	0·2586	0·5375	0·6938	0·8702	1·079	1·350	1·771	2·160	2·650	3·012	3·852	4·221
14	0·2582	0·5366	0·6924	0·8681	1·076	1·345	1·761	2·145	2·624	2·977	3·787	4·140
15	0·2579	0·5357	0·6912	0·8662	1·074	1·341	1·753	2·131	2·602	2·947	3·733	4·073
16	0·2576	0·5350	0·6901	0·8647	1·071	1·337	1·746	2·120	2·583	2·921	3·686	4·015
17	0·2573	0·5344	0·6892	0·8633	1·069	1·333	1·740	2·110	2·567	2·898	3·646	3·965
18	0·2571	0·5338	0·6884	0·8620	1·067	1·330	1·734	2·101	2·552	2·878	3·610	3·922
19	0·2569	0·5333	0·6876	0·8610	1·066	1·328	1·729	2·093	2·539	2·861	3·579	3·883
20	0·2567	0·5329	0·6870	0·8600	1·064	1·325	1·725	2·086	2·528	2·845	3·552	3·850
21	0·2566	0·5325	0·6864	0·8591	1·063	1·323	1·721	2·080	2·518	2·831	3·527	3·819
22	0·2564	0·5321	0·6858	0·8583	1·061	1·321	1·717	2·074	2·508	2·819	3·505	3·792
23	0·2563	0·5317	0·6853	0·8575	1·060	1·319	1·714	2·069	2·500	2·807	3·485	3·768
24	0·2562	0·5314	0·6848	0·8569	1·059	1·318	1·711	2·064	2·492	2·797	3·467	3·745
25	0·2561	0·5312	0·6844	0·8562	1·058	1·316	1·708	2·060	2·485	2·787	3·450	3·725
26	0·2560	0·5309	0·6840	0·8557	1·058	1·315	1·706	2·056	2·479	2·779	3·435	3·707
27	0·2559	0·5306	0·6837	0·8551	1·057	1·314	1·703	2·052	2·473	2·771	3·421	3·690
28	0·2558	0·5304	0·6834	0·8546	1·056	1·313	1·701	2·048	2·467	2·763	3·408	3·674
29	0·2557	0·5302	0·6830	0·8542	1·055	1·311	1·699	2·045	2·462	2·756	3·396	3·659
30	0·2556	0·5300	0·6828	0·8538	1·055	1·310	1·697	2·042	2·457	2·750	3·385	3·646
32	0·2555	0·5297	0·6822	0·8530	1·054	1·309	1·694	2·037	2·449	2·738	3·365	3·622
34	0·2553	0·5294	0·6818	0·8523	1·052	1·307	1·691	2·032	2·441	2·728	3·348	3·601
36	0·2552	0·5291	0·6814	0·8517	1·052	1·306	1·688	2·028	2·434	2·719	3·333	3·582
38	0·2551	0·5288	0·6810	0·8512	1·051	1·304	1·686	2·024	2·429	2·712	3·319	3·566
40	0·2550	0·5286	0·6807	0·8507	1·050	1·303	1·684	2·021	2·423	2·704	3·307	3·551
50	0·2547	0·5278	0·6794	0·8489	1·047	1·299	1·676	2·009	2·403	2·678	3·261	3·496
60	0·2545	0·5272	0·6786	0·8477	1·045	1·296	1·671	2·000	2·390	2·660	3·232	3·460
120	0·2539	0·5258	0·6765	0·8446	1·041	1·289	1·658	1·980	2·358	2·617	3·160	3·373
∞	0·2533	0·5244	0·6745	0·8416	1·036	1·282	1·645	1·960	2·326	2·576	3·090	3·291